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Transient thermal contact resistance

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Abstract—The short-time asymptotic solution for the heat flow between two similar bodies placed in contact has recently been given by Barber. The long-time asymptotic solution is given here: it has exactly the same form as the short-time asymptote

$$\frac{Q}{KT_0} = C_0 + \frac{C_1}{\sqrt{(\pi\alpha t)}}$$

but, in general, has different coefficients.

INTRODUCTION

BARBER [1] has recently shown that when two semi-infinite solids of the same material at different, uniform, temperatures are placed in contact, the heat flow between them at small times is given by

$$\frac{Q}{KT_0} = \frac{A}{\sqrt{(\pi\alpha t)}} + \tfrac{1}{2}S \tag{1}$$

where T_0 is half the initial temperature difference, K and α the thermal conductivity and diffusivity, respectively, and A and S the area and periphery of the contact area.

Regarding this as the flow from an area A maintained at T_0 into a semi-infinite solid initially at zero, we see that a related heat flow problem is the flow from a hemispherical surface A to the solid, for which the temperature distribution is (see Section 9.10 in Carslaw and Jaeger [2])

$$\theta = T_0 \frac{a}{r} \operatorname{erfc} \frac{r-a}{2\sqrt{(\alpha t)}} \tag{2}$$

where a is the radius of the hemisphere. Heasley [3] has used this model as an approximation to the flow from a circular contact area, following its use by Holm [4] to estimate electrical contact resistance.

From equation (2) the heat flow from the hemisphere is

$$\frac{Q}{KT_0} = 2\pi a + \frac{2\pi a^2}{\sqrt{(\pi\alpha t)}} \tag{3a}$$

$$= S + \frac{A}{\sqrt{(\pi\alpha t)}} \tag{3b}$$

in Barber's terms: that is, for this case the short-time transient solution (differing by a factor 0.5 from Barber's)† is exact at all times.

† The 0.5 need cause no dismay, since Barber's arguments are directed at a *plane* contact area.

Now, it seems clear that for *any* case of flow into a semi-infinite solid the flow rate will ultimately behave as

$$\frac{Q}{KT_0} \sim C_0 + \frac{C_1}{\sqrt{(\pi\alpha t)}} \tag{4}$$

(the $\pi\alpha$ is inserted for later convenience). That is, it has the same form as the short-time solution, although in general the coefficients will be different. For many contact geometries we know the value of C_0 (if R_0 is the steady-state thermal contact resistance then $C_0 = (KR_0)^{-1}$), and it is certainly not directly related to the periphery of the contact, though it often varies in a similar way. For example, for an elliptical contact area we have

$$\frac{\text{conductance } C_0}{\text{semi-periphery}} = \frac{\pi}{K(e)E(e)} \quad \text{where } e^2 = 1 - b^2/a^2 \tag{5}$$

($K(e)$ and $E(e)$ are complete elliptic integrals). This falls slowly from $4/\pi$ for a circle to 1.1 when $e^2 = 0.9$, but then quickly falls to zero as $e \to 1$. The corresponding values of C_1 are not known, though the value for a circular contact may be deduced from the long-time temperature distribution found by Norminton and Blackwell [5] to be $C_1 = 8b^2/\pi$. We show below that quite generally, there is a simple relation between C_1 and the steady thermal contact conductance C_0.

THEOREM

It may easily be verified that

$$\theta = \frac{F}{2\pi K r t^{1/2}} \exp\left(-\frac{r^2}{4\alpha t}\right) \tag{6}$$

gives the temperature due to a point source of varying intensity $F/t^{1/2}$ on the surface of a semi-infinite solid.

When $4\alpha t \gg r^2$, this reduces to

NOMENCLATURE

- a radius of hemispherical contact, semi-major axis of ellipse
- A area of contact
- b radius of circular contact, semi-minor axis of ellipse
- C_0 steady-state thermal conductance
- C_1 coefficient: see equation (5)
- e eccentricity of elliptical contact
- F coefficient of variable heat input rate, $F/t^{1/2}$
- K thermal conductivity
- q, q_0 heat input rate/unit area
- Q heat flow rate
- Q_0 constant heat flow rate
- r radial distance
- R_0 steady-state thermal resistance
- R_0^* R_0K
- S periphery of contact area
- t time
- T_0 (constant) temperature of contact area
- $T(t)$ (variable) temperature at contact.

Greek symbols

- α thermal diffusivity
- θ temperature
- μ, μ' unknown multipliers.

$$\theta = \frac{F}{2\pi Krt^{1/2}} + O(t^{-3/2}) \tag{7}$$

and the leading term is precisely the result

$$\theta = \frac{q}{2\pi Kr}$$

giving the steady temperature due to a constant heat source q: that is, the same equation is valid with $q = \text{const.}$ or with $q = F/t^{1/2}$.

It follows that *we can immediately write down the leading term in the transient solution for any distribution of surface heating varying as $t^{-1/2}$ from the corresponding steady-state solution.*

CIRCULAR CONTACT

For a circular contact of radius b, it is well known that a constant heat input $q = q_0(1 - r^2/b^2)^{-1/2}$ gives a uniform temperature $T_0 = Q_0/4bK$ over $r < b$, where Q_0 is the total rate of heat input $2\pi q_0 b^2$. It has been shown [6] that the approach to this steady temperature is given by

$$T_0 - T(t) = \frac{Q_0}{2\pi K} \cdot \frac{1}{\sqrt{(\pi\alpha t)}} + O(t^{-3/2}) \tag{8}$$

independent of position or of the details of the source.

From the theorem above, we know that a time-varying heat input $\mu q_0(1 - r^2/b^2)^{-1/2}t^{-1/2}$ will also give a uniform temperature over $r < b$, equal to

$$T(t) = \frac{\mu Q_0}{4bK} \cdot \frac{1}{t^{1/2}} + O(t^{-3/2}).$$

Thus, a heat input $Q_0(1 + \mu/t^{1/2})$ will give

$$T(t) = \frac{Q_0}{4bK} - \frac{Q_0}{2\pi K} \cdot \frac{1}{\sqrt{(\pi\alpha t)}} + \frac{\mu Q_0}{4bKt^{1/2}} + O(t^{-3/2})$$

which will be constant (to $O(t^{-3/2})$) if we take $\mu = 2b/(\pi\sqrt{(\pi\alpha)})$. Reversing the argument, we deduce that a constant surface temperature T_0 gives a heat flow

$$\frac{Q}{Kt_0} = 4b\left(1 + \frac{2b}{\pi\sqrt{(\pi\alpha t)}}\right) \tag{9}†$$

i.e. $C_0 = 4b = (2/\pi)S$ and $C_1 = 8b^2/\pi = (8/\pi^2)A$.

We note that both coefficients in the long-time solution are close to the values in Barber's short-term solution.

Figure 1 compares the two asymptotic expansions, valid for long times and short times, respectively. Clearly either asymptote gives a fair approximation over the whole range, though each gives a 25% error if used at the 'wrong' end. Figure 1 also shows the equation representing the numerical solution by Schneider *et al.* [7], i.e.

$$\frac{4bKT_0}{Q} = 0.43 \tanh\left[0.37 \ln\left(\frac{4\alpha t}{b^2}\right)\right] + 0.57. \tag{10}$$

Although this agrees well with the asymptotic solutions in the middle of the range, it is clearly in conflict as $t \to 0$ or $t \to \infty$.

CONTACT OF ANY SHAPE

It is easy to extend the above analysis to a contact of arbitrary shape. We define the (semi-non-dimensional) steady-state resistance R_0^* by $KT_0 = Q_0R_0^*$ and assume that R_0^* is known. Then a steady heat input Q_0 will give

$$T(t) = \frac{Q_0R_0^*}{K} - \frac{Q_0}{2\pi K} \cdot \frac{1}{\sqrt{(\pi\alpha t)}} + O(t^{-3/2})$$

using equation (9) again; and a heat input $\mu'Q_0/\sqrt{(\pi\alpha t)}$ distributed in the same way as Q_0 will give a temperature $\mu'Q_0R_0^*/(K\sqrt{(\pi\alpha t)})$. Then taking $\mu'R_0^* = 1/2\pi$ we deduce that for a constant temperature T_0 the heat flow will be given by

† Differentiating Norminton and Blackwell's temperature distribution leads to this result, *and* to a zero coefficient for the next $(t^{-3/2})$ term.

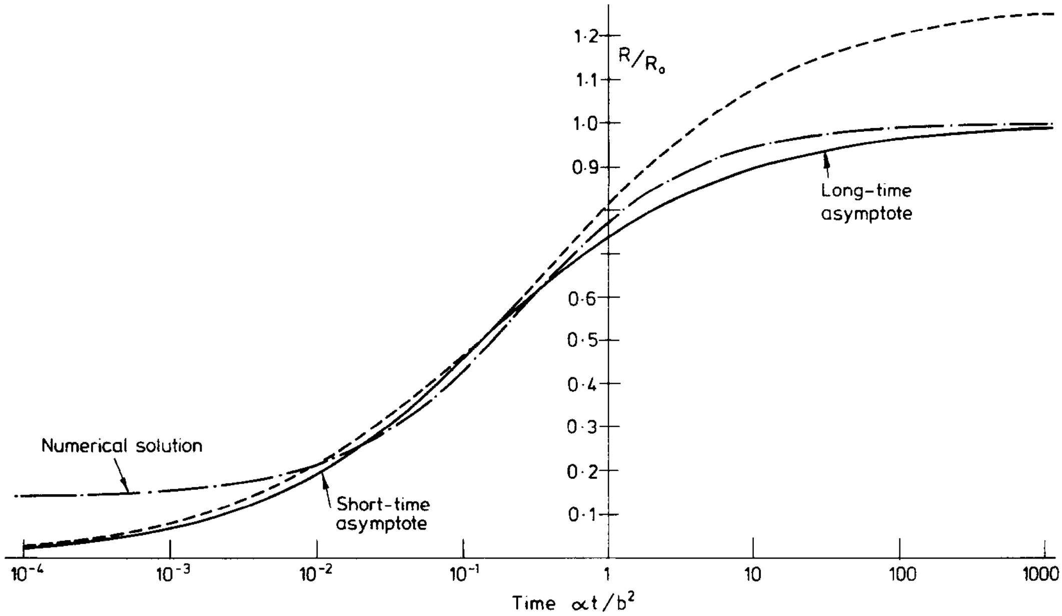

FIG. 1.

$$\frac{Q}{KT_0} = \frac{1}{R_0^*} + \frac{1}{2\pi R_0^{*2}} \cdot \frac{1}{\sqrt{(\pi\alpha t)}} + O(t^{-3/2}) \quad (11)$$

i.e. quite generally

$$C_1 = 1/(2\pi R_0^{*2}) = C_0^2/2\pi. \quad (12)$$

For the hemispherical 'contact' we have $R_0^* = 1/2\pi a$ and note with satisfaction† that C_1 becomes $2\pi a^2$ as in equation (3a). For an elliptical contact we have $R_0^* = K(e)/2\pi a$ so that $C_1 = 2\pi a^2/(K(e))^2$; and we see that C_1 deviates considerably from the short-term value A as soon as the contact becomes appreciably elliptical.

WARNING

Nothing in the above argument appears to rely on there being only a *single* contact area (except perhaps the condition $4\alpha t \gg r^2$). Yet it would seem that when contact occurs at two equal, well-separated individual contacts, the values of both C_0 and C_1 are doubled, and $C_1 \neq C_0^2/2\pi$. The reader will no doubt be able to resolve this paradox; the author's speculations are given in the Appendix.

CONCLUSION

The transient heat flow between two solids at long contact times becomes

$$\frac{Q}{KT_0} = \frac{1}{R_0^*} + \frac{1}{2\pi R_0^{*2}} \cdot \frac{1}{\sqrt{(\pi\alpha t)}}$$

where R_0^*/K is the steady-state thermal resistance. The time constants *may* be close to the values in the short-time solution

$$\frac{Q}{KT_0} = \tfrac{1}{2}S + \frac{A}{\sqrt{(\pi\alpha t)}}$$

but may deviate considerably from them.

† And some surprise, since our proof is for a plane heat source.

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APPENDIX. TWO EQUAL, WELL-SEPARATED CONTACT AREAS

It would appear that two well-separated contact areas should act independently, so that if C_0 is the conductance of an individual area, then for the two

$$\frac{Q}{KT_0} = 2C_0 + \frac{C_0^2}{\pi\sqrt{(\pi\alpha t)}} \quad (A1)$$

whereas the theory of the paper gives

$$\frac{Q}{KT_0} = 2C_0 + \frac{2C_0^2}{\pi\sqrt{(\pi\alpha t)}}. \quad \text{(A2)}$$

One may argue that if b is a characteristic dimension of the individual contact, while d is their separation, then equation (A1) holds for $b^2 \ll 4\alpha t \ll d^2$ and equation (A2) for $4\alpha t \gg d^2$. However, this fails to explain why the co-operation of the two contacts *increases* the heat flow: surely the heat flow should be reduced?

In fact we know that the steady-state heat flow will be reduced below $2C_0$ because of the 'mutual resistance' of the two contacts (see ref. [8]): the temperature at one contact is raised by the heat flow through the second, and this reduces the flow through the first. To be definite, consider two circular contacts of radius b so that $C_0 = 4b$: then we have

$$T_0 = \frac{Q_1}{4Kb} + \frac{Q_2}{2\pi Kd}$$

where the heat flows Q_1 and Q_2 through the two contacts are each $\frac{1}{2}Q_0$. Then

$$\frac{KT_0}{Q} = \frac{1}{2}\left(\frac{1}{4b} + \frac{1}{2\pi d}\right)$$

and equation (A2) should be corrected to

$$\frac{Q}{KT_0} \sim \frac{2C_0}{1+2b/\pi d} + \frac{2C_0^2}{\pi\sqrt{(\pi\alpha t)}}. \quad \text{(A3)}$$

The change compared with equation (A1) is

$$-\frac{4C_0 b}{\pi d} + \frac{C_0^2}{\pi\sqrt{(\pi\alpha t)}} \sim \frac{C_0^2}{\pi}\left(\frac{1}{\sqrt{(\pi\alpha t)}} - \frac{1}{d}\right)$$

and, provided $\pi\alpha t > d^2$, this is a *decrease* as expected.

RESISTANCE THERMIQUE VARIABLE DE CONTACT

Résumé—La solution asymptotique de court temps pour le transfert de chaleur entre deux corps semblables placés en contact a été récemment étudiée par Barber. On donne ici la solution asymptotique à long terme: elle a exactement la même forme que celle à court temps

$$\frac{Q}{KT_0} = C_0 + \frac{C_1}{\sqrt{(\pi\alpha t)}}$$

mais, en général, avec des coefficients différents.

KONTAKTWIDERSTAND UNTER INSTATIONÄREN BEDINGUNGEN

Zusammenfassung—Die asymptotische Kurzzeitlösung für den Wärmestrom an der Kontaktfläche zwischen zwei ähnlichen Körpern wurde vor kurzem von Barber vorgestellt. Hier wird nun die asymptotische Langzeitlösung beschrieben, welche exakt die gleiche Form wie die Kurzzeitlösung besitzt:

$$\frac{Q}{KT_0} = C_0 + \frac{C_1}{\sqrt{(\pi\alpha t)}}$$

aber im allgemeinen andere Koeffizienten aufweist.

ТЕПЛОВОЕ КОНТАКТНОЕ СОПРОТИВЛЕНИЕ В НЕСТАЦИОНАРНОМ СЛУЧАЕ

Аннотация—Недавно Барбером было получено асимптотическое решение для теплового потока между двумя идентичными телами, приведенными в контакт, при малых временах. В данной статье приводится асимптотическое решение для больших времен, которое имеет тот же вид, что и асимптота для малых времен

$$\frac{Q}{kT_0} = C_0 + \frac{C_1}{\sqrt{(\pi\alpha t)}}$$

но с другими коэффициентами.